The period of the simple pendulum to second order in the amplitude

Note on session S1, 24 November 2026

Abstract

The question was the coefficient c_2 of the θ_0^2 correction to the small-oscillation period $T_0 = 2\pi\sqrt{L/g}$ of the simple pendulum, derived analytically and checked against a numerical evaluation of the exact period. The session finds $c_2 = 1/16$, the classical value, from the exact period (a complete elliptic integral) and again from a Poincaré–Lindstedt expansion. An independent computation evaluates the exact period at 13 amplitudes from 0.01 to 3 rad and recovers c_2 from the numerical data alone. The next coefficient is $c_4 = 11/3072$. The second-order formula underestimates the period at every amplitude tested: its relative error stays below 0.4% up to 1 rad and reaches 39% at 3 rad.

Setting

The pendulum is a point mass on a massless rigid rod of length L, moving in a vertical plane under uniform gravity g, without friction. The angle $\theta(t)$ of the rod from the downward vertical, in radians, obeys

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0, \qquad \omega_0 = \sqrt{g/L}, \qquad T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{L/g}. \tag{1}$$

The mass drops out of the period. The amplitude $\theta_0 \in (0, \pi)$ is the largest angle reached, so the pendulum never goes over the top. The exact period $T(\theta_0)$ is expanded as $T = T_0\,(1 + c_2\,\theta_0^2 + c_4\,\theta_0^4 + \dots)$. The complete elliptic integral of the first kind is written with the modulus k,

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}, \tag{2}$$

and every series of K in this note is a series in k. SciPy and mpmath take instead the parameter $m = k^2$ as argument; the computation below passes them $m = \sin^2(\theta_0/2)$. These are the conventions of `conventions.md`.

1 The second-order coefficient

Result 1 (Second-order coefficient of the period)**.** *For the simple pendulum at amplitude* θ_0*,*

$$T(\theta_0) = T_0\left(1 + \frac{\theta_0^2}{16} + O(\theta_0^4)\right), \qquad T_0 = 2\pi\sqrt{L/g}, \tag{3}$$

so $c_2 = 1/16$.[1] [CL-S1-1]

This is the classical value; the session reproduces and checks it.

1.1 Derivation

Derivation D-S1-1 starts from energy conservation, $\dot\theta^2 = 2\omega_0^2(\cos\theta - \cos\theta_0)$. The four quarter swings of one period take equal times, by the symmetries $\theta \to -\theta$ and $t \to -t$, so

$$T(\theta_0) = \frac{4}{\omega_0}\int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}. \tag{4}$$

With $k = \sin(\theta_0/2)$ and the substitution $\sin(\theta/2) = k\sin\phi$, this becomes the exact period

$$T(\theta_0) = 4\sqrt{\frac{L}{g}}\,K\big(\sin(\theta_0/2)\big) = \frac{2T_0}{\pi}\,K(k). \tag{5}$$

Expanding the integrand of K by the binomial series and integrating term by term (Wallis integrals) gives, for $0 \le k < 1$,

$$K(k) = \frac{\pi}{2}\sum_{n\ge 0}\left[\binom{2n}{n}\frac{1}{4^n}\right]^2 k^{2n} = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + O(k^6)\right). \tag{6}$$

Inserting $k^2 = \sin^2(\theta_0/2) = \theta_0^2/4 - \theta_0^4/48 + O(\theta_0^6)$ and collecting powers of θ_0,

$$\frac{T(\theta_0)}{T_0} = 1 + \frac{\theta_0^2}{16} + \frac{11\,\theta_0^4}{3072} + O(\theta_0^6). \tag{7}$$

The coefficient $c_2 = \frac{1}{4}\cdot\frac{1}{4}$ comes from the first correction in K alone. The next one, $c_4 = -\frac{1}{4}\cdot\frac{1}{48} + \frac{9}{64}\cdot\frac{1}{16} = 11/3072$, also involves the quartic term of

[1] Derivation D-S1-1 and the independent computation C-S1-2, each confirmed by a blind review.

$\sin^2(\theta_0/2)$. The expansion converges near $\theta_0 = 0$, since K is analytic in k^2 for $|k^2| < 1$. The sign of c_2 is the expected one: $\sin\theta < \theta$, so the restoring torque is weaker than in the linear problem and larger swings are slower.

The derivation also obtains c_2 without the elliptic integral, by a Poincaré–Lindstedt expansion of (1) with $\sin\theta \approx \theta - \theta^3/6$. Removing the resonant term at order θ_0^2 fixes the frequency, $\omega^2 = \omega_0^2\,(1 - \theta_0^2/8 + O(\theta_0^4))$, hence $T = 2\pi/\omega = T_0\,(1 + \theta_0^2/16 + O(\theta_0^4))$.

A blind review confirmed the derivation. It reproduced the series (7) symbolically, evaluated the period integral by quadrature and by direct integration of the equation of motion, and checked the Poincaré–Lindstedt step.

1.2 Check against the exact period

Computation C-S1-2 was set up without access to the derivation. Symbolically, it rederives the exact period (5) and expands it to order θ_0^8, and it carries a Poincaré–Lindstedt expansion to order θ_0^6. The two routes agree exactly: $c_2 = 1/16$, $c_4 = 11/3072$ and $c_6 = 173/737280$. Numerically, it evaluates the exact ratio $T/T_0 = (2/\pi)K(\sin(\theta_0/2))$ at 13 amplitudes from 0.01 to 3 rad with mpmath (40 digits), with SciPy and with an arithmetic–geometric-mean evaluation of K; the three agree to within 1.3×10^{-15}. A direct numerical integration of (1), which uses no elliptic function, agrees with them to within 1.3×10^{-14}.

The residual $T/T_0 - 1 - \theta_0^2/16$ scales as $\theta_0^{4.0014}$ on $[0.01, 0.3]$ rad; at $\theta_0 = 0.01$, the residual divided by θ_0^4 is 0.0035807526, against $c_4 = 0.0035807292$, and the gap is $c_6\,\theta_0^2$. A least-squares fit of the integration data alone, on $[0.05, 0.5]$ rad, gives $c_2 = 0.0625000033$; the error of 3×10^{-9} comes from truncating the fit, not from the integration.

A blind review confirmed the computation. It re-ran the script and reproduced its output exactly, and it recovered $c_2 = 1/16$ by routes the script does not use: the hypergeometric form $(2/\pi)K = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$, a direct expansion of the period integral, a 50-digit quadrature and a different ODE solver. A three-point Richardson extrapolation of the quadrature at 0.001, 0.002 and 0.004 rad gives 1/16 to 20 digits.

1.3 Accuracy of the second-order formula

Table 1 compares the exact period with the second-order formula. The formula underestimates the period at every amplitude tested. Its relative error stays below 4×10^{-3} up to 1 rad (57°), and grows to 6% at 2 rad

and 39% at 3 rad. Near $\theta_0 = \pi$ the exact period diverges logarithmically, $T/T_0 \approx (2/\pi)\ln\bigl(4/\cos(\theta_0/2)\bigr)$, which no polynomial in θ_0 can follow.

θ_0 (rad)	θ_0 (deg)	T/T_0, exact	$1+\theta_0^2/16$	relative error
0.1	5.7	1.0006254	1.000625	-3.6×10^{-7}
0.5	28.6	1.0158525	1.015625	-2.2×10^{-4}
1.0	57.3	1.0663342	1.0625	-3.6×10^{-3}
1.5	85.9	1.1619687	1.140625	-1.8×10^{-2}
2.0	114.6	1.3289045	1.25	-5.9×10^{-2}
2.5	143.2	1.6429824	1.390625	-0.15
3.0	171.9	2.5712339	1.5625	-0.39

Table 1: Exact period and second-order formula, from computation C-S1-2. The relative error is $[T_0(1+\theta_0^2/16)-T]/T$.

2 What did not work

Nothing failed in this session. One error was anticipated and ruled out: SciPy and mpmath take the parameter $m = k^2$, so passing them the modulus $k = \sin(\theta_0/2)$ would give a wrong period. The computation passes $m = \sin^2(\theta_0/2)$, and the direct integration of the equation of motion, which uses no elliptic function, agrees with the library values to 10^{-14}.

3 Open question and next steps

The goal of the project is answered: $c_2 = 1/16$, derived by two analytic routes and checked against the exact period at 13 amplitudes. The evidence is robust, since a derivation and an independent computation each passed a blind review. The result stays a working claim until the lead copies it into the draft. The project lists no next step, so the direction is for the lead to set. Two continuations need little work. The exact period (5) and $c_4 = 11/3072$ could get claim cards of their own, so that they can enter the draft as results; both artefacts and both reviews already check them. A short literature survey could supply a reference for this classical result, since the workspace has no paper card and this note cites nothing.