A classical reference for the period of the pendulum to second order

Note on session S2, 24 November 2026

Abstract

The session looked for one citable reference for $T = T_0\,(1 + \theta_0^2/16)$, Result 1 of the draft[CL-S1-1], which so far cites nothing. Lima and Arun (2006, Am. J. Phys.) state the exact period and this formula, which they attribute to Bernoulli (1749). Gatland (2007, a preprint) gives the full series of the exact period. No source disputes $c_2 = 1/16$. The session adds no new claim; it suggests citing Lima and Arun next to Result 1.

Setting

Notation is that of `conventions.md`: amplitude θ_0, $T_0 = 2\pi\sqrt{L/g}$, $T = T_0\,(1 + c_2\,\theta_0^2 + c_4\,\theta_0^4 + \dots)$, and $K(k)$ written with the modulus $k = \sin(\theta_0/2)$, as in both papers (Gatland calls it x). The session made no derivation or computation: what follows is what the papers state, as read in the survey, and the numerical checks quoted were made during the survey, outside any reviewed artefact.

1 What the two papers state

Lima and Arun [2] derive from energy conservation the exact period

$$T = 4\sqrt{L/g}\;K(k) = \frac{2T_0}{\pi}\,K(k), \qquad k = \sin(\theta_0/2), \tag{1}$$

the form found in S1, and state without deriving it

$$T_2 = T_0\left(1 + \frac{\theta_0^2}{16}\right), \tag{2}$$

"found by Bernoulli in 1749 from a perturbative analysis".[1] Their stated accuracy of (2), a relative error above 0.1% beyond 41°, matches the survey's own evaluation. The attribution to Bernoulli is second-hand: they take it from a 1960 textbook by C. J. Smith, which the session did not read. A citation should therefore attribute the formula to Bernoulli "according to Lima and Arun".

Gatland [1] expands the same exact period as

$$\frac{T}{T_0} = \sum_{n\geq 0} a_n\, k^{2n} = 1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots \tag{3}$$

with $a_0 = 1$ and $a_n = (1 - 1/2n)^2\, a_{n-1}$ (we write a_n for his coefficients, to keep c_n for ours). He does not expand in θ_0; substituting $k^2 = \theta_0^2/4 - \theta_0^4/48 + \dots$ gives $T/T_0 = 1 + \theta_0^2/16 + 11\,\theta_0^4/3072 + \dots$, the two coefficients found in S1. His Table I lists the exact T/T_0: 1.004301 (15°), 1.017409 (30°), 1.039973 (45°), 1.073182 (60°), 1.118959 (75°) and 1.180341 (90°). The survey reproduced all six with `mpmath.ellipk`, called with the parameter $m = k^2$. The paper is marked "Submitted to Amer. J. Phys.", and the survey found no evidence that it was published: it is a good working reference, and a weaker citation than Lima and Arun.

Both papers agree with Result 1 of the draft[CL-S1-1], and so do the closed-form large-amplitude approximations they compare (Kidd–Fogg, Molina and others), which the survey checked all reduce to $1+\theta_0^2/16$ at second order.

2 What did not work

Textbooks cannot be cited in the workspace: Landau and Lifshitz's *Mechanics*, Thornton and Marion's *Classical Dynamics* and the NIST DLMF (§22.19(i) and Eq. 19.5.1, read online by the survey) have no arXiv or INSPIRE record, so they cannot have a paper card or a `references.bib` entry. The session therefore settled on a journal paper.

3 Open question and next steps

The goal of `project.md`, answered in S1, now has a citable source, and the lead asked to stop after this survey. The immediate step is to cite Lima and Arun [2] next to Result 1 of the draft, attributing (2) to Bernoulli according

[1] Their Eqs. 5 and 11 in the arXiv version, which the survey read in its LaTeX source; the published numbering may differ.

to them. If the project continues, Gatland's table offers an external check of the S1 computation, and Amore and Aranda (arXiv:math-ph/0303052, noted by the survey but not carded) are said to derive $\omega = \omega_0 \, (1 - \theta_0^2/16)$ by the Poincaré–Lindstedt method, the second route of S1.

References

[1] Ian R. Gatland. Systematic approximations for the period of a finite amplitude pendulum, 2007. arXiv:0707.1029.

[2] F. M. S. Lima and P. Arun. An accurate formula for the period of a simple pendulum oscillating beyond the small angle regime. *Am. J. Phys. 74, 892*, 2006. arXiv:physics/0510206.